Answer all the questions.   Max Marks: 50

1. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. For $x \in (a, b)$ and $h > 0$ define the difference quotient

$$D_h f(x) = \frac{f(x + h) - f(x)}{h}, \tag{DQ}$$

   show that there exists a constant $C > 0$ such that

$$|D_h f(x) - f'(x)| \leq Ch$$

   where $'$ denotes differentiation. Assuming f to be complex with $x, h \in \mathbb{R}$, show that

$$C_h f(x) = \frac{Im \; f(x + ih)}{h} \tag{CQ}$$

   is another approximation to $f'(x)$. Let $x_0 \in (a, b)$ and $f(x) = 1/x$ write down the analytical expressions for $D_h f(x_0)$ and $C_h f(x_0)$. Which of the formulas (DQ or CQ) would be your choice for computing the numerical derivative of the above mentioned function on the computer? [15]

2. Define a matrix norm and explain how it differs from a vector norm . Show that the vector norm
$$\|x\|_\infty = \max_i |x_i|$$

   for $x \in \mathbb{R}^n$ induces the matrix norm   [10]

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_j |a_{ij}|.$$

3. Let $\boldsymbol{A}$ be the matrix
$$\begin{bmatrix} \epsilon & 0 & 0 & 1 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 1 & 0 & 0 & \epsilon \end{bmatrix}$$

   where $\epsilon \neq 0$. Write down the $\boldsymbol{LU}$ decomposition of this matrix. [15]

4. We wish to solve the equation
$$\sin(x) = x^2.$$

   One solution is at $x = 0$, but we are interested in finding the other solution $x^\star \neq 0$ Write down an iteration formula for Newton's method for solving this problem, i.e an expression for x_k in terms of x_{k-1} and your choice for x_0 so as to get $x^\star$. [10]